Vacuum Energies of Non–Abelian String–Configurations in 3+1 Dimensions

H. Weigel[a)], M. Quandt[b)], N. Graham[c)], O. Schröder[d)]

[a)] *Physics Department, Stellenbosch University, Matieland 7602, South Africa*
[b)] *Institute for Theoretical Physics, Tübingen University, D–72076 Tübingen, Germany*
[c)] *Department of Physics, Middlebury College, Middlebury, VT 05753, USA*
[d)] *science+computing ag, Hagellocher Weg 73, D–72070 Tübingen, Germany*

We develop a method to compute the fermion contribution to the vacuum polarization energy of string–like configurations in a non–abelian gauge theory. This calculation has been hampered previously by a number of technical obstacles. We use gauge invariance of the energy and separation of length scales in the energy density to overcome these obstacles. We present a proof–of–principle investigation that shows that this energy is small in the $\overline{\rm MS}$ renormalization scheme. The generalization to other schemes is straightforward.

I. INTRODUCTION

Various field theories suggest the existence of string–like configurations, which are the particle physics analogues of vortices or magnetic flux tubes in condensed matter physics. Often they are called cosmic strings to distinguish them from the fundamental variables in string theory and to indicate that they stretch over cosmic length scales. If they exist, cosmic strings, can potentially have significant cosmological effects. We refer to ref. [1] for a recent review on the physical implications of strings in the standard model and beyond.

In the standard model of particle physics, string solutions [2–4] are not topologically stable and thus can only be stabilized dynamically. In exploring the existence of cosmic strings, it is therefore important to be able to accurately calculate their energies. Here we will develop and apply a method to compute the fermion contribution to the leading order quantum correction to the energy, the so–called vacuum polarization or Casimir energy. In a large N_C scenario with many internal degrees of freedom, the fermion contribution dominates that of the bosons.

A number of previous studies have investigated quantum properties of string and vortex configurations, invoking either approximations, simplified configurations, or lower dimensions to cope with technical difficulties. Naculich [5] has shown that in the limit of weak coupling, fermion fluctuations destabilize the non–abelian Z–string. The quantum properties of Z–strings have also been connected to non–perturbative anomalies [6]. The fermionic vacuum polarization energies of QED flux–tubes, and abelian flux–tubes more generally, were investigated using heat–kernel methods [7, 8], world line numerics [9] as well as the phase shift method [10]. The heat–kernel method was also used to study self–dual vortices [11]. This method and the world line approach limit renormalization to the subtraction of the divergences in the heat–kernel expansion, thereby obscuring the connection to perturbative renormalization. As we will explain later, the phase shift approach is capable of making straightforward contact with any renormalization condition that is formulated in terms of (momentum space) Green's functions. In lower dimensions the ultra–violet divergences are less severe, which made the computation feasible for the case of two spatial dimensions [12], while the obstacles that arise for the physical case became soon obvious [13]. A first attempt at a full calculation of the quantum corrections to the Z–string energy was carried out in ref. [14]. Those authors were only able to compare the energies of two string configurations, rather than comparing a single string configuration to the vacuum; these limitations arise from subtleties of the renormalization process that we address in this paper. Also, the contribution of bosonic fluctuations to the vacuum polarization energies have been estimated for vortex configurations using the heat–kernel method [15] and string backgrounds within the phase shift approach [16], which

we will use here for the fermionic fluctuations. Stability of cosmic string currents was considered in [17].

We begin by expressing the vacuum polarization energy as the renormalized sum over (half) the change of the single particle energies caused by the localized background. Previously, the fermionic vacuum polarization energy of strings has been computed for the case of two spatial dimensions [12]. The main purpose of the present paper is to extend this computation to the physical case of three spatial dimensions, yielding the vacuum polarization energy per unit length of the string. Though that extension seems straightforward since the string is translationally invariant with respect to this additional coordinate, a number of obstacles arise. They are mainly related to the more complex structure of ultra–violet divergences.

It is well established that the vacuum polarization energies of extended background field configurations, such as solitons or vortices, are unambiguously obtained from a momentum integral that involves the derivative of the phase shifts in the potential generated by the background field [18]. These phase shifts measure the distortion of the spectrum of quantum fluctuations caused by the background. If an object is translationally invariant with respect to a subset of the coordinates, we can use the phase shifts calculated in the nontrivial dimensions combined with appropriate kinematic coefficients to describe the full spectrum of fluctuations [19]. These coefficients vary only with the number of trivial dimensions, but not with the background field. In addition to integrating the result over the magnitude of momentum k, we also need to sum over angular momentum channels. It has previously been shown that for string–type configurations, these two operations are not absolutely convergent, and inappropriately exchanging them may yield an unphysical convergence [13]. In $D = 2 + 1$ spacetime dimensions, the problem is mitigated because the relevant Feynman diagram is manifestly finite. In $D = 3 + 1$, the complicated structure of the divergent third and fourth order Feynman diagrams makes the exchange unavoidable. This problem is not specific to the string, and can only be avoided by analytic continuation to complex momenta. This procedure requires a careful construction of the Jost function for the Dirac scattering problem. On the other hand, the integration over imaginary momenta automatically includes the bound state contribution to the vacuum polarization energy [18, 20].

A second, more severe obstacle is specific to the string configuration. The method for computing the vacuum polarization energy also requires the evaluation of Feynman diagrams whose external legs are given in terms of Fourier transforms of the background field. In this calculation, the leading terms of the Born series are subtracted and the corresponding contributions to the vacuum polarization energy are added back in as (renormalized) Feynman diagrams. We stress that the energy is not computed from this perturbation expansion; rather, the advantage of our approach is that this expansion is used to make contact with standard procedures of perturbative renormalization, which in particular allows the implementation of any renormalization scheme that is formulated for the momentum space Green's functions of the theory. The problem that arises for the string is that the configuration does not vanish at spatial infinity, and so its Fourier transform is ill–defined. While gauge–invariant combinations of the Higgs and vector fields do vanish at spatial infinity, the Feynman series is not gauge invariant term by term. To circumvent this problem, we need to introduce a *return string*, localized at a some distance ρ_0 that is large compared to the typical extension w of the physical string. This return string unwinds the physical string so that the Fourier transforms and thus the individual Feynman diagrams give well–defined functionals. Even though the vacuum polarization energy $E_{\rm vac}$ is a non–local functional, for $\rho_0 \gg w$ there should be a separation of scales,

$$E_{\rm vac} = E_{\rm ps}(w) + E_{\rm rs}(\rho_0) \tag{1}$$

for the energies of the physical string ($E_{\rm ps}$) and that of the return string ($E_{\rm rs}$), since the overlap between the associated densities vanishes. Then computing $E_{\rm vac}$ and finding $E_{\rm rs}(\rho_0)$ for a special

case, yields $E_{\rm ps}(w)$, the quantity that we are seeking. A similar approach has been previously used to compute the vacuum polarization energy of electromagnetic flux tubes. In that case dimensional analysis indicated that $\lim_{\rho_0\to\infty} E_{\rm rs}(\rho_0) = 0$, which was then confirmed by directly calculating the energy density [10]. Here the situation is more complicated, due to the presence of the Higgs field. A naïve analysis indicates that in general $E_{\rm rs}(\rho_0)$ actually grows quadratically with ρ_0. If the Higgs field of the return string stays on the chiral circle (meaning that its magnitude is fixed at its vacuum expectation value, so that only its isospin orientation changes in space), $E_{\rm rs}(\rho_0)$ remains constant. However, there is no *a priori* reason for it to vanish. As an aside we note that, in contrast to the QED case, the classical energy of the return string does not vanish as $\rho_0 \to \infty$. So we need to find the energy of the return string, which then is subtracted from eq. (1) to separate the energy of the physical string. Unfortunately, the general return string background induces potentials in the scattering problem that behave like $1/\rho^2$ at small distances. Though the corresponding scattering matrix can still be computed, the evaluation of the Jost function remains obscure because the bounds needed to prove the analytic structure from the iterative solution are no longer satisfied. One way out of the dilemma would be to compute the return string energy within a covariant expansion that is reliable for large ρ_0 [21], but we have a better option. At least in a particular gauge (and we have to choose one anyhow to do the calculation) the small distance singularity disappears for return strings that have support only on the chiral circle. As a result we can calculate $E_{\rm rs}$ in this case, and numerically we find that it vanishes as $\rho_0 \to \infty$ in the $\overline{\rm MS}$ renormalization scheme. This is not in contradiction to the general analysis that yields a constant, because the non–zero term is omitted in $\overline{\rm MS}$.

Another technical obstacle that did not emerge in the case of two space dimensions is the increased computational effort needed. In the case of three space dimensions, the third and fourth order Feynman diagrams induce logarithmic divergences. When folded with the Fourier transforms of the background potentials, these diagrams become higher–dimensional integrals that are cumbersome to treat numerically. We therefore introduce a *fake boson* field whose second order Feynman diagram possesses the identical logarithmic divergence [22, 23]. Since it is a solution to a well–defined scattering problem, we are ensured of the analytic properties of the resulting Jost function and we can deal with it on equal footing as the one from the Dirac problem.

In this paper we have collected the ingredients for computing the vacuum polarization energy of the string. For simplicity we use the $\overline{\rm MS}$ renormalization scheme and leave the use of physical (on–shell) conditions and dynamical questions such as the stabilization of the string by bound fermions to a forthcoming paper.

To be specific, we consider a left–handed $SU(2)$ gauge theory in which a fermion doublet Ψ is coupled to the gauge field W_μ and the Higgs field ϕ. Since we are studying only fermionic fluctuations, we can ignore the self-interactions of the Higgs and gauge fields and simply assume the Higgs potential is minimized at a nonzero vacuum expectation value $|\phi| = v$. We express the Higgs doublet $\phi = \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix}$ as the matrix

$$\Phi = \begin{pmatrix} \phi_0^* & \phi_+ \\ -\phi_+^* & \phi_0 \end{pmatrix} . \tag{2}$$

We then have the interaction Lagrangian

$$\mathcal{L} = \overline{\Psi} i\gamma_\mu D^\mu P_L \Psi + \overline{\Psi} i\gamma_\mu \partial^\mu P_R \Psi - f\,\overline{\Psi}\left(\Phi P_R + \Phi^\dagger P_L\right)\Psi\,, \tag{3}$$

where $P_{R,L} = \frac{1}{2}\left(1 \pm \gamma_5\right)$ are projection operators on right– and left–handed components, respectively. The Higgs–fermion interaction is parameterized by the Yukawa coupling constant f, while

the gauge coupling constant is defined via the covariant derivative

$$D^\mu = \partial^\mu - igW^\mu \,. \tag{4}$$

Note that the vector fields are 2×2 matrices in (weak) isospace. To simplify the model, we have taken the Weinberg angle to be zero, so that there is no photon coupling, and assumed that the fermion doublet is degenerate in mass.

Finally we list the counterterms that are required to compute the vacuum polarization energy,

$$\mathcal{L}_{\rm ct} = c_1 {\rm tr}_I \left[W_{\mu\nu} W^{\mu\nu}\right] + \frac{c_2}{2} {\rm tr}_I \left[\left(D_\mu \Phi\right)^\dagger \left(D^\mu \Phi\right)\right] + \frac{c_3}{2} {\rm tr}_I \left[\Phi^\dagger \Phi - v^2\right] + \frac{c_4}{4} \left({\rm tr}_I \left[\Phi^\dagger \Phi - v^2\right]\right)^2 \,, \tag{5}$$

where the gauge field tensor is $W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - ig\left[W_\mu, W_\nu\right]$ for the vector field. As indicated, the traces are with respect to isospin. Of course, the counterterms are just the typical gauge–invariant terms appearing in the classical Lagrangian for the Higgs and gauge fields. Note also that the c_3 and c_4 counterterms cancel quadratic ultra–violet divergences, while c_1 and c_2 cancel logarithmic divergences. In a model with only two spatial dimensions, c_1 and c_2 would be finite, though not necessarily zero [12].

The remainder of the paper is organized as follows. In the next section we introduce the string configuration and discuss the obstacles mentioned above in more detail. In section III we briefly explain the phase shift method for computing the vacuum polarization energy. In particular we discuss the interface method for the case when the background configuration is translationally invariant in one coordinate. We also introduce the fake boson technique for simplifying the higher–order divergences. In section IV we explain the use of the return string on the chiral circle. Numerical results are presented in section V, and concluding remarks and an outlook for applications are given in section VI.

II. THE Z–STRING

The string configuration only depends on the distance ρ from the symmetry axis (which we choose to be $\hat{z}$) and the corresponding azimuthal angle φ. It is characterized by a non–vanishing angular dependence at spatial infinity. In matrix notation we write in temporal gauge ($W^0 = 0$)

$$\begin{aligned} \vec{W} &= n \sin(\xi_1) \frac{f_G(\rho)}{\rho} \hat{\varphi} \begin{pmatrix} \sin(\xi_1) & i\cos(\xi_1)\, {\rm e}^{-in\varphi} \\ -i\cos(\xi_1)\, {\rm e}^{in\varphi} & -\sin(\xi_1) \end{pmatrix} \qquad \text{and} \\ \Phi &= v f_H(\rho) \begin{pmatrix} \sin(\xi_1)\, {\rm e}^{-in\varphi} & -i\cos(\xi_1) \\ -i\cos(\xi_1) & \sin(\xi_1)\, {\rm e}^{in\varphi} \end{pmatrix} . \end{aligned} \tag{6}$$

This configuration is commonly called a Z–string, because the corresponding component $Z \propto {\rm tr}_I(W\tau_3)$ exhibits the spatial dependence of an abelian string. The radial functions $f_G(\rho)$ and $f_H(\rho)$ approach unity at spatial infinity while they vanish at the origin ($\rho = 0$). They are the typical profiles of the Nielson–Oleson string [24]. We parameterize these profile functions via

$$f_H(\rho) = 1 - {\rm e}^{-\frac{\rho}{w_H}} \qquad \text{and} \qquad f_G(\rho) = 1 - {\rm e}^{-\left(\frac{\rho}{w_G}\right)^2} . \tag{7}$$

Then the fields $\vec{W}$ and Φ are $\mathcal{O}(\rho)$ as $\rho \to 0$ and ambiguities resulting from an ill–defined azimuthal angle do not arise. We have also introduced a general winding number n for the string, though in the numerical calculation we will only consider $n = 1$. In ref. [12] we treated the angle $0 \leq \xi_1 \leq \frac{\pi}{2}$ as a variational parameter of the string configuration. We emphasize that ξ_1 describes the orientation of the Higgs field on the chiral circle. It will prove to be very useful to introduce the return string by allowing it to be space dependent such that it vanishes at spatial infinity.

A. String in the Dirac Equation

The Dirac equation that arises from the Lagrangian is dealt with in two steps. The dependence on the z–coordinate in which the string is invariant separates in a pure phase factor $e^{-ip_z z}$. In the second step we are thus left with a two–dimensional problem[1]

$$\begin{aligned} H \;=\; -i \begin{pmatrix} 0 & \vec{\sigma}\cdot\hat{\rho} \\ \vec{\sigma}\cdot\hat{\rho} & 0 \end{pmatrix} \partial_\rho - \frac{i}{\rho} \begin{pmatrix} 0 & \vec{\sigma}\cdot\hat{\varphi} \\ \vec{\sigma}\cdot\hat{\varphi} & 0 \end{pmatrix} \partial_\varphi + \frac{ns}{2\rho} f_G \begin{pmatrix} -\vec{\sigma}\cdot\hat{\varphi} & \vec{\sigma}\cdot\hat{\varphi} \\ \vec{\sigma}\cdot\hat{\varphi} & -\vec{\sigma}\cdot\hat{\varphi} \end{pmatrix} I_G \\ + m f_H \left[c \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + s \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} i I_P \right], \end{aligned} \tag{8}$$

where $s = \sin(\xi_1)$ and $c = \cos(\xi_1)$. The explicit matrices in the above equation refer to the spinor indices and σ_i are the 2×2 Pauli matrices. Furthermore we have introduced the isospin matrices

$$I_G(\varphi) = \begin{pmatrix} -s & -ic\,\mathrm{e}^{in\varphi} \\ ic\,\mathrm{e}^{-in\varphi} & s \end{pmatrix}_I \qquad \text{and} \qquad I_P(\varphi) = \begin{pmatrix} 0 & \mathrm{e}^{in\varphi} \\ \mathrm{e}^{-in\varphi} & 0 \end{pmatrix}_I . \tag{9}$$

If E is an eigenvalue of H, then the single particle energy is $\pm\sqrt{E^2 + p_z^2}$. Calculating the vacuum polarization energy requires us to integrate over p_z in the framework of the interface formalism, which we review below. The sign degeneracy arises from the anti–commutator $\{H, \gamma_3\} = 0$.

It is straightforward to extract the fermion–string interaction from eq. (8)

$$H_{\mathrm{int}} = \frac{ns}{2\rho} f_G \begin{pmatrix} -\vec{\sigma}\cdot\hat{\varphi} & \vec{\sigma}\cdot\hat{\varphi} \\ \vec{\sigma}\cdot\hat{\varphi} & -\vec{\sigma}\cdot\hat{\varphi} \end{pmatrix} I_G + m \left[(c f_H - 1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + s f_H \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} i I_P \right], \tag{10}$$

In view of the asymptotic behavior of the radial functions $f_{G,H}$ mentioned above, many of the problems with computing the vacuum polarization energy become immediately obvious. The interaction does not vanish at spatial infinity, but rather approaches a pure gauge configuration. This implies that there is no straightforward way to set up a Born series, whose necessity we will recognize in the following section. Furthermore we cannot compute any Fourier transformation of the interaction, which makes it impossible to use Feynman diagrams to impose conventional renormalization schemes such as on–shell or $\overline{\mathrm{MS}}$.

It is tempting to perform a (singular) gauge transformation

$$\widetilde{H} = U^\dagger H U \qquad \text{with} \qquad U = \begin{pmatrix} s\,\mathrm{e}^{-in\varphi} & -ic \\ -ic & s\,\mathrm{e}^{in\varphi} \end{pmatrix}_I P_L + P_R \tag{11}$$

such that

$$\widetilde{H}_{\mathrm{int}} = \frac{ns}{2\rho} (f_G - 1) \begin{pmatrix} -\vec{\sigma}\cdot\hat{\varphi} & \vec{\sigma}\cdot\hat{\varphi} \\ \vec{\sigma}\cdot\hat{\varphi} & -\vec{\sigma}\cdot\hat{\varphi} \end{pmatrix} I_G + m (f_H - 1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{12}$$

However, this only shifts the problem to the origin. As $\lim_{\rho\to 0} f_G = 0$ the $\frac{1}{\rho}$ term induces a $\frac{1}{\rho^2}$ potential in the corresponding second order Schrödinger type scattering problem. This is outside the standard regime in which analyticity of the Jost function is established [25]. We comment on the use of $\widetilde{H}_{\mathrm{int}}$ at the end of section V.

[1] We utilize the standard representation for the Dirac–matrices and perform a global chiral rotation $U = -\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} P_L + P_R$ to restore the typical form of the Dirac equation.

B. Unwinding the String

To set up the scattering problem we must find a way to treat eq. (10). A possible solution to that problem can be envisaged immediately. If the angle ξ_1 were not constant, but a radial function going from its value at the origin (which defines the configuration of interest) to zero at some large distance ρ_0, the scattering problem would be treatable without altering the magnitude of the Higgs field. This configuration corresponds to a return string on the chiral circle that has less severe ultra–violet singularities, since the c_3 and c_4 counterterms in eq. (5) vanish on the chiral circle. For convenience we parameterize $\sin\xi_1$ rather than ξ_1 itself,

$$s(\rho) = \sin(\xi_1)\,\frac{1-\tanh\left(w_0\,\frac{\rho-\rho_0}{\rho_0}\right)}{1+\tanh(w_0)} \qquad \text{and} \qquad c(\rho) = \sqrt{1-s^2(\rho)}\,. \tag{13}$$

Of course, the chiral circle condition continues to hold. This addition does not alter the form of the Dirac Hamiltonian, eq. (10). However, it must be taken into account that the isospin matrix I_G now has radial dependence. The combination of eqs. (7) and (13) then describes the physical string for $\rho \ll \rho_0$ and a non–interacting theory for $\rho \gg \rho_0$. To accomplish the separation of scales indicated in eq. (1) we require $\rho_0 \gg w_{G,H}$. With these prerequisites, the Hamiltonian, eq. (8), has a well–defined scattering problem that we will now describe.

C. Scattering off the Unwound String

As a first step we introduce grand–spin type states that couple spin and isospin, to account for the angular dependence. For fixed angular momentum ℓ there are four of them,

$$\begin{aligned}
\langle\varphi; SI|\ell++\rangle &= \mathrm{e}^{i(\ell+n)\varphi}\begin{pmatrix}1\\0\end{pmatrix}_S \otimes \begin{pmatrix}1\\0\end{pmatrix}_I & \langle\varphi; SI|\ell+-\rangle &= -i\,\mathrm{e}^{i\ell\varphi}\begin{pmatrix}1\\0\end{pmatrix}_S \otimes \begin{pmatrix}0\\1\end{pmatrix}_I \\
\langle\varphi; SI|\ell-+\rangle &= i\,\mathrm{e}^{i(\ell+n+1)\varphi}\begin{pmatrix}0\\1\end{pmatrix}_S \otimes \begin{pmatrix}1\\0\end{pmatrix}_I & \langle\varphi; SI|\ell--\rangle &= \mathrm{e}^{i(\ell+1)\varphi}\begin{pmatrix}0\\1\end{pmatrix}_S \otimes \begin{pmatrix}0\\1\end{pmatrix}_I .
\end{aligned} \tag{14}$$

where S and I refer to the spin and isospin subspaces, respectively. These grand–spin states serve to construct the Dirac spinors in coordinate space,

$$\begin{aligned}
\langle\rho|++\rangle &= \begin{pmatrix} f_1(\rho)|\ell++\rangle \\ g_1(\rho)|\ell-+\rangle\end{pmatrix} & \langle\rho|+-\rangle &= \begin{pmatrix} f_2(\rho)|\ell+-\rangle \\ g_2(\rho)|\ell--\rangle\end{pmatrix} \\
\langle\rho|-+\rangle &= \begin{pmatrix} f_3(\rho)|\ell-+\rangle \\ g_3(\rho)|\ell++\rangle\end{pmatrix} & \langle\rho|--\rangle &= \begin{pmatrix} f_4(\rho)|\ell--\rangle \\ g_4(\rho)|\ell+-\rangle\end{pmatrix} ,
\end{aligned} \tag{15}$$

where we have suppressed the angular momentum index of the radial functions because the Dirac equation is diagonal in this quantum number. We combine these eight radial functions into two vectors

$$\vec{f} = \begin{pmatrix} f_1(\rho)\\ f_2(\rho)\\ f_3(\rho)\\ f_4(\rho)\end{pmatrix} \qquad \text{and} \qquad \vec{g} = \begin{pmatrix} g_1(\rho)\\ g_2(\rho)\\ g_3(\rho)\\ g_4(\rho)\end{pmatrix} \tag{16}$$

to write the Dirac equation as a set of eight coupled first order linear differential equations in the matrix form

$$(E-m)\,\vec{f} \;=\; V_{uu}\,\vec{f} + (D_u + V_{ud})\;\vec{g}$$

$$(E+m)\,\vec{g} \;=\; (D_d + V_{du})\;\vec{f} + V_{dd}\,\vec{g}\,. \tag{17}$$

The derivative operators are fully contained in the diagonal matrices

$$\begin{aligned} D_u &= \mathrm{diag}\left(\partial_\rho + \frac{\ell+n+1}{\rho}, \partial_\rho + \frac{\ell+1}{\rho}, -\partial_\rho + \frac{\ell+n}{\rho}, -\partial_\rho + \frac{\ell}{\rho}\right) \\ D_d &= \mathrm{diag}\left(-\partial_\rho + \frac{\ell+n}{\rho}, -\partial_\rho + \frac{\ell}{\rho}, \partial_\rho + \frac{\ell+n+1}{\rho}, \partial_\rho + \frac{\ell+1}{\rho}\right)\,. \end{aligned} \tag{18}$$

We will give the explicit form of the real 4×4 matrices V_i in terms of the radial functions when we set up the Born series for the scattering data. Here it suffices to note that these matrices are real and vanish at spatial infinity, so the asymptotic solutions are cylindrical Bessel and Hankel functions. In particular, the Hankel functions

$$\begin{aligned} \mathcal{H}_u &= \mathrm{diag}\left(H^{(1)}_{\ell+n}(k\rho), H^{(1)}_{\ell}(k\rho), H^{(1)}_{\ell+n+1}(k\rho), H^{(1)}_{\ell+1}(k\rho)\right) \\ \mathcal{H}_d &= \mathrm{diag}\left(H^{(1)}_{\ell+n+1}(k\rho), H^{(1)}_{\ell+1}(k\rho), H^{(1)}_{\ell+n}(k\rho), H^{(1)}_{\ell}(k\rho)\right) \end{aligned} \tag{19}$$

that parameterize the outgoing asymptotic fields with (radial) momentum k can be used to set up the scattering problem via the matrix generalization

$$\vec{f} \quad\longrightarrow\quad \mathcal{F}\cdot\mathcal{H}_u \qquad \text{and} \qquad \vec{g} \quad\longrightarrow\quad \kappa\,\mathcal{G}\cdot\mathcal{H}_d\,. \tag{20}$$

Note that $\kappa = \mathrm{sgn}(E)\,\sqrt{\frac{E-m}{E+m}}$ is well defined for either sign of the energy eigenvalue since $|E|>m$ for the scattering solution. Using the dispersion relation for real momenta $E^2 = k^2+m^2$, we may also write $\kappa = \frac{k}{E+m} = \frac{E-m}{k}$. The boundary conditions for the 4×4 complex matrices $\mathcal{F}$ and $\mathcal{G}$ are simply $\lim_{\rho\to\infty}\mathcal{F} = \mathbb{1}$ and $\lim_{\rho\to\infty}\mathcal{G} = \mathbb{1}$ so the various columns of the above products refer to outgoing waves in different channels. Demanding finally that the scattering wave–functions are regular at the origin defines the scattering matrix. It can be obtained from either the upper or lower components

$$\begin{aligned} \mathcal{S} &= -\lim_{\rho\to 0}\,\mathcal{H}_u^{-1}\cdot\mathcal{F}^{-1}\cdot\mathcal{F}^*\cdot\mathcal{H}_u^* \\ \mathcal{S} &= -\lim_{\rho\to 0}\,\mathcal{H}_d^{-1}\cdot\mathcal{G}^{-1}\cdot\mathcal{G}^*\cdot\mathcal{H}_d^*\,. \end{aligned} \tag{21}$$

This construction ensures that the physical scattering solution,[2]

$$\Psi^{(u)}_{\text{s.c.}} = \mathcal{F}^*\cdot\mathcal{H}_u^* + \mathcal{G}\cdot\mathcal{H}_u\cdot\mathcal{S}\,, \qquad \Psi^{(d)}_{\text{s.c.}} = \mathcal{G}^*\cdot\mathcal{H}_u^* + \mathcal{F}\cdot\mathcal{H}_u\cdot\mathcal{S}\,. \tag{22}$$

is regular at the origin, $\rho\to 0$. An important numerical check (besides unitarity) is that both equations yield identical scattering matrices. We write the Dirac equation for the matrices $\mathcal{F}$ and $\mathcal{G}$ in a form that will later simplify the Born series,

$$\begin{aligned} \partial_\rho\mathcal{F} &= \left[\overline{\mathcal{M}}_{ff} + O_d\right]\cdot\mathcal{F} + \mathcal{F}\cdot\mathcal{M}^{(r)}_{ff} + \left[\overline{\mathcal{M}}_{fg} + kC\right]\cdot\mathcal{G}\cdot Z_d \\ \partial_\rho\mathcal{G} &= \left[\overline{\mathcal{M}}_{gg} + O_u\right]\cdot\mathcal{G} + \mathcal{G}\cdot\mathcal{M}^{(r)}_{gg} + \left[\overline{\mathcal{M}}_{gf} - kC\right]\cdot\mathcal{F}\cdot Z_u\,. \end{aligned} \tag{23}$$

The matrices

$$C \;=\; \mathrm{diag}(-1,-1,1,1)\,,$$

[2] Since the matrices V_i are real, $\mathcal{F}^*\cdot\mathcal{H}_u^*$ and $\mathcal{G}^*\cdot\mathcal{H}_d^*$ also solve the Dirac equation.

$$
\begin{aligned}
O_u &= \tfrac{1}{\rho}\,\mathrm{diag}\,(-(\ell+n+1),-(\ell+1),\ell+n,\ell)\,,\\
O_d &= \tfrac{1}{\rho}\,\mathrm{diag}\,(\ell+n,\ell,-(\ell+n+1),-(\ell+1))\,,\\
Z_d &= \mathrm{diag}\left(\frac{H^{(1)}_{\ell+n+1}(k\rho)}{H^{(1)}_{\ell+n}(k\rho)}\,,\frac{H^{(1)}_{\ell+1}(k\rho)}{H^{(1)}_{\ell}(k\rho)}\,,\frac{H^{(1)}_{\ell+n}(k\rho)}{H^{(1)}_{\ell+n+1}(k\rho)}\,,\frac{H^{(1)}_{\ell}(k\rho)}{H^{(1)}_{\ell+1}(k\rho)}\right)=(Z_u)^{-1}\,,\\
\mathcal{M}^{(r)}_{gg} &= kC\cdot Z_u - O_u \quad\text{and}\quad \mathcal{M}^{(r)}_{ff} = -kC\cdot Z_d - O_d\,,
\end{aligned}
\tag{24}
$$

are purely kinematic, while the background fields are contained in the real–valued 4×4 matrices

$$
\begin{aligned}
\overline{\mathcal{M}}_{gg} &= \begin{pmatrix}\alpha_G R_+ & \alpha_P R_-\\ -\alpha_P R_- & -\alpha_G R_+\end{pmatrix} \qquad \overline{\mathcal{M}}_{gf} = \frac{1}{\kappa}\left[\alpha_H C + \alpha_G\begin{pmatrix}0 & -R_+\\ R_+ & 0\end{pmatrix}\right]\\
\overline{\mathcal{M}}_{ff} &= \begin{pmatrix}-\alpha_G R_+ & \alpha_P R_-\\ -\alpha_P R_- & \alpha_G R_+\end{pmatrix} \qquad \overline{\mathcal{M}}_{fg} = \kappa\left[\alpha_H C - \alpha_G\begin{pmatrix}0 & -R_+\\ R_+ & 0\end{pmatrix}\right],
\end{aligned}
\tag{25}
$$

where

$$
R_+ = \begin{pmatrix}s & c\\ c & -s\end{pmatrix} \qquad R_- = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \tag{26}
$$

The profile functions enter via

$$
\alpha_H = m\,(cf_H - 1)\,, \quad \alpha_P = msf_H \quad\text{and}\quad \alpha_G = \frac{ns}{2\rho}\,f_G\,. \tag{27}
$$

Note that the inclusion of the return string makes R_+ a function of ρ, while R_- remains constant in space.

The Born series is a straightforward but tedious expansion in the matrices $\overline{\mathcal{M}}_i$, which are linear in the background fields. For example, at first order we have to solve

$$
\begin{aligned}
\partial_\rho \mathcal{F}_1 &= O_d\cdot\mathcal{F}_1 + \mathcal{F}_1\cdot\mathcal{M}^{(r)}_{ff} + kC\cdot\mathcal{G}_1\cdot Z_d + \overline{\mathcal{M}}_{ff} + \overline{\mathcal{M}}_{fg}\cdot Z_d\\
\partial_\rho \mathcal{G}_1 &= O_u\cdot\mathcal{G}_1 + \mathcal{G}_1\cdot\mathcal{M}^{(r)}_{gg} - kC\cdot\mathcal{F}_1\cdot Z_u + \overline{\mathcal{M}}_{gg} + \overline{\mathcal{M}}_{gf}\cdot Z_u\,,
\end{aligned}
\tag{28}
$$

with the asymptotic conditions $\lim_{\rho\to\infty}\mathcal{F}_1 = 0$ and $\lim_{\rho\to\infty}\mathcal{G}_1 = 0$. The first order correction to the scattering matrix then becomes $\mathcal{S}_1 = \lim_{\rho\to 0}\mathcal{H}_u^{-1}\,[\mathcal{F}_1 - \mathcal{F}_1^*]\,\mathcal{H}_u^*$. The solutions $\mathcal{F}_1$ and $\mathcal{G}_1$ are the sources for the second order terms $\mathcal{F}_2$ and $\mathcal{G}_2$, yielding the second Born approximation $\mathcal{S}_2 = \lim_{\rho\to 0}\mathcal{H}_u^{-1}\,[\mathcal{F}_1\,(\mathcal{F}_1^* - \mathcal{F}_1) + \mathcal{F}_2 - \mathcal{F}_2^*]\,\mathcal{H}_u^*$. This iteration in the background fields can be repeated to any desired order, yielding the Born series for the scattering matrix $\mathcal{S} = \mathbb{1} + \mathcal{S}_1 + \mathcal{S}_2 + \ldots$.

D. Jost Function

Ultimately we need to compute the phase of $\det(\mathcal{S})$, which according to eq. (21) is related to the phase of $F = F_\ell(k) = \det\left(\lim_{\rho\to 0}\mathcal{F}\right)$, or equivalently $G = G_\ell(k) = \det(\lim_{\rho\to 0}\mathcal{G})$. Furthermore, we need to sum over angular momentum ℓ and integrate over radial momentum k after subtracting sufficiently many terms of the Born series. This procedure is not only numerically cumbersome because these are oscillating functions of k, but it may even cause erroneous results [13] since these sums/integrals are not absolutely convergent. These obstacles are avoided by analytically continuing to imaginary momenta $t = ik$ and performing the integrals along the branch cut $t > m$. The analytic continuation for the Dirac equation is conceptually different from the well–studied

Schrödinger case because $E = \pm\sqrt{k^2+m^2}$ causes the complex momentum plane to have two sheets. So on the real axis we have to pick one sign, continue to complex momenta and compute the Jost function on the imaginary axis. This procedure must then be repeated for the other sign and then all discontinuities must be collected at the end. In the present problem we are fortunate because the solutions to the Dirac equation exhibit charge conjugation symmetry along the real axis. Therefore $\det(\mathcal{S})$ does not change under $E \to -E$ and there is no additional discontinuity in the Jost function. Moreover, the Jost function is real on the imaginary axis, as in the Schrödinger problem. However, the way this comes about in the string problem requires us to be careful when constructing the Jost function for complex momenta.

We describe the case with $E = \sqrt{k^2+m^2}$. The analytic continuation concerns the Hankel functions, which turn into modified Bessel functions: $Z_u \to Y_u$ and $Z_d \to Y_d$, with

$$Y_u = \mathrm{diag}\left(\frac{K_{\ell+n}(t\rho)}{K_{\ell+n+1}(t\rho)}, \frac{K_{\ell}(t\rho)}{K_{\ell+1}(t\rho)}, -\frac{K_{\ell+n+1}(t\rho)}{K_{\ell+n}(t\rho)}, -\frac{K_{\ell+1}(t\rho)}{K_{\ell}(t\rho)}\right) = -\left(Y_d\right)^{-1} . \tag{29}$$

Furthermore we have the change of the kinematic coefficient $\kappa \to z_\kappa$, with

$$z_\kappa = \frac{m + i\sqrt{t^2-m^2}}{t} , \tag{30}$$

a pure phase. The differential equations, which we solve subject to the boundary conditions that $\mathcal{F}$ and $\mathcal{G}$ approach unity at spatial infinity, read

$$\begin{aligned} \partial_\rho \mathcal{F} &= \left[\overline{\mathcal{M}}_{ff} + O_d\right]\cdot\mathcal{F} + \mathcal{F}\cdot\mathcal{M}^{(r)}_{ff} + \left[\overline{\mathcal{M}}_{fg} - tC\right]\cdot\mathcal{G}\cdot Y_d \\ \partial_\rho \mathcal{G} &= \left[\overline{\mathcal{M}}_{gg} + O_u\right]\cdot\mathcal{G} + \mathcal{G}\cdot\mathcal{M}^{(r)}_{gg} + \left[\overline{\mathcal{M}}_{gf} + tC\right]\cdot\mathcal{F}\cdot Y_u . \end{aligned} \tag{31}$$

For simplicity we have omitted the momentum arguments in the radial wave–functions $\mathcal{F}$ and $\mathcal{G}$. It should be apparent from the context that these are distinct from the solutions to eqs. (23). The coefficient matrices are slightly modified:

$$\begin{aligned} \mathcal{M}^{(r)}_{gg} &= -tC\cdot Y_u - O_u & \mathcal{M}^{(r)}_{ff} &= tC\cdot Y_d - O_d \\ \overline{\mathcal{M}}_{gf} &= z_\kappa\left[\alpha_H C + \alpha_G\begin{pmatrix} 0 & -R_+ \\ R_+ & 0\end{pmatrix}\right] & \overline{\mathcal{M}}_{fg} &= -z^*_\kappa\left[\alpha_H C - \alpha_G\begin{pmatrix} 0 & -R_+ \\ R_+ & 0\end{pmatrix}\right] , \end{aligned} \tag{32}$$

while $\overline{\mathcal{M}}_{gg}$ and $\overline{\mathcal{M}}_{ff}$ are as on the real axis. Note, that in contrast to the Schrödinger problem, the differential equations do not become real on the imaginary axis. Rather, charge conjugation $E \to -E$ induces complex conjugation. It is therefore not surprising that the naïve continuation to $\lim_{\rho\to 0}\mathcal{F}$ is not real. Instead, we find numerically that $F = G^*$. Interestingly enough, for a given value of t, the imaginary part is the same for all angular momenta. The origin for this imaginary part lies in the subtle definition of the Jost function via the Wronskian between the Jost solution, *i.e.* $\mathcal{F}$ or $\mathcal{G}$, and the regular solution that satisfies momentum independent boundary conditions at the origin. The latter are required to ensure the regular solution to be an analytic function. Analyticity of the Jost solution is guaranteed by the non–singular behavior of the interaction potentials in the presence of the return string. At the origin, the Higgs field differs from its vacuum expectation value (it actually vanishes). In the Dirac case this modifies the relative weight of the upper and lower components. On the real axis this weight is given by κ in eq. (20). More precisely, at the origin the non–diagonal elements of the matrices in eq. (32) vanish and the eight differential equations decouple with respect to the index on the radial functions in eq. (16), *i.e.* spin and weak isospin. For real momenta a typical solution in the vicinity of $\rho = 0$ then looks like [8]

$$\begin{pmatrix} f_4 \\ g_4\end{pmatrix} \sim \left(\frac{k}{q}\right)^l \begin{pmatrix} \sqrt{E+mcf_H(0)}\, J_l(q\rho) \\ \\ \sqrt{E-mcf_H(0)}\, J_{l+1}(q\rho)\end{pmatrix} \tag{33}$$

with $q = \sqrt{E^2 - (mcf_H(0))^2}$ and similar dependencies for the other six radial functions. The square–root coefficients cause the proper definition of the Jost function, $\nu(t)$, to be

$$\exp\left[\nu(t)\right] = \left(\frac{\tau - im}{\tau - imcf_H(0)}\right)^2 \lim_{\rho\to 0}\det(\mathcal{F}) = \left(\frac{\tau + im}{\tau + imcf_H(0)}\right)^2 \lim_{\rho\to 0}\det(\mathcal{G}) \tag{34}$$

with $\tau = \sqrt{t^2 - m^2}$. The power of two occurs because we compute the determinant of a 4×4 matrix. Note that this redefinition not only cancels the imaginary parts, but also modifies the real part. Furthermore it cancels the logarithmic singularity in $\ln\lim_{\rho\to 0}\det(\mathcal{F})$ observed numerically at $t \sim m$. Since f_H is part of the interaction, this correction factor also undergoes expansion in the framework of the Born series. Otherwise, *i.e.* for $\det(\mathcal{F})$, the Born series is constructed as for real momenta by iterating the differential eq. (31) in $\overline{\mathcal{M}}_i$.

The resulting Jost function is a continuous function in the upper complex momentum plane and the branch cuts in the Dirac equation do not carry over to ν. This is a consequence of charge conjugation invariance in the present model.

III. PHASE SHIFT APPROACH IN D=3+1

In this section we collect all the pieces needed to compute the vacuum polarization energy for the configuration in which the physical and the return strings are combined.

A. Interface Formalism

The interface formalism addresses the problem of computing the vacuum polarization energy for configurations that are translationally invariant in a subset of coordinates. As laid out in ref. [19], this approach only requires scattering data from the lower dimensional space in which the configuration is non–trivial. However, we still have to adapt the formalism to the present string problem and, in particular, motivate the above recipe for computing the logarithmic Jost function $\nu(t)$ on the imaginary momentum axis.

Our starting point is the interface formula for the vacuum polarization energy per unit length of a configuration that is translationally invariant in one coordinate,[3]

$$\begin{aligned}\Delta E_\delta^{(N)} \;=\; \frac{1}{4\pi}\sum_\ell \Bigg\{ D_\ell \int_0^\infty \frac{dk}{\pi}\left[(k^2+m^2)\ln\left(\frac{k^2+m^2}{\mu^2}\right) - k^2\right] \frac{d}{dk}\left[\delta_\ell(k)\right]_N \\ + \sum_j \left[(\epsilon_{j,\ell})^2 \ln\frac{(\epsilon_{j,\ell})^2}{\mu^2} - (\epsilon_{j,\ell})^2 + m^2\right]\Bigg\}.\end{aligned} \tag{35}$$

A few remarks on the notation are in order. The most important input is the phase shift $\delta_\ell = \ln\det(\mathcal{S})$ of the scattering matrix in eq. (21). Using the factorization property of the determinant and the fact that $Y_\nu(z) \gg J_\nu(z)$ for $z \to 0$ we may write

$$\delta_\ell(k) = \frac{1}{i}\ln\det\lim_{\rho\to 0}\mathcal{F}_\ell(\rho,k)^{-1}\mathcal{F}_\ell^*(\rho,k) = \frac{1}{i}\ln\det\lim_{\rho\to 0}\mathcal{G}_\ell(\rho,k)^{-1}\mathcal{G}_\ell^*(\rho,k)\,. \tag{36}$$

The square bracket with index N in eq. (35) indicates that the first N terms of the Born series have been subtracted. This is important both for the integral to converge and also to ensure that the

[3] In comparison with eq.(6) of ref. [19] a factor two emerged from the two signs of the single particle energies.

semi–circle at infinity does not contribute when performing the integral by contour integration in the complex momentum plane. Later we will add back this contribution in terms of renormalized Feynman diagrams. This is indeed the heart of the phase shift approach, as it allows the implementation of standard (perturbative) renormalization conditions. In case of the string background the degeneracy factor D_ℓ in eq. (35) is

$$D_\ell = \begin{cases} 1\,, & \ell = -n \\ 2\,, & \text{otherwise}\,, \end{cases} \tag{37}$$

where $n = 1$ is the winding number of the scalar field introduced in the string background eq. (6). Finally $\epsilon_{j,\ell}$ denote the bound state energies in the ℓ^{th} partial wave with $\epsilon_{j,\ell} < m$. The renormalization scale μ has no effect as a consequence of sum rules for scattering data [26]. We nevertheless require it for dimensional reasons and for simplicity take $\mu = m$.

To set up the contour integral we first remark that [*cf.* eq. (24)] $Z_d(-k) = -Z_d^*(k)$. This implies that[4] $\mathcal{F}_\ell(\rho, -k) = \mathcal{F}_\ell^*(\rho, k)$ since the boundary condition for $\mathcal{F}$ at $\rho \to \infty$ is independent from k. (Identical relations hold, of course, for $\mathcal{G}$.) We therefore write for the momentum integral in eq. (35) (with $E = \sqrt{k^2 + m^2}$)

$$\begin{aligned} &\int_0^\infty \frac{dk}{\pi} \left[E^2 \ln\left(\frac{E^2}{m^2}\right) - k^2\right] \frac{d}{dk} \left[\delta_\ell(k)\right]_N = \\ &\quad - \int_{-\infty}^\infty \frac{dk}{2\pi i} \left[E^2 \ln\left(\frac{E^2}{m^2}\right) - k^2\right] \frac{d}{dk} \left[\ln \det \mathcal{F}_\ell(0,k)\right]_N = \\ &\quad - \int_{-\infty}^\infty \frac{dk}{2\pi i} \left[E^2 \ln\left(\frac{E^2}{m^2}\right) - k^2\right] \frac{d}{dk} \left[\ln\left(\frac{E+m}{E+mcf_H(0)}\right)^2 + \ln \det \mathcal{F}_\ell(0,k)\right]_N \end{aligned} \tag{38}$$

where the argument $\rho = 0$ is understood as the limit $\rho \to 0$. The additional term introduced in the second equation adds a piece to the integrand that is odd in k and therfore does not change the integral. This modification has introduced the Jost function, since $\nu(t)$ is precisely the analytic continuation of the quantity in square brackets. We have already argued that $\nu(t)$ is an analytic function in the upper half momentum plane. Hence when closing the contour, we only need to consider the discontinuity of the logarithm along the cut $t \geq m$, which is $2\pi i$. In addition we need to add the contributions from poles to the contour integral. As in the non–relativistic case, the existence of any bound state is related to a simple root of the Jost function at $k = i\kappa_{j,\ell} = i\sqrt{m^2 - (\epsilon_{j,\ell})^2}$. Therefore

$$\text{tr}\left[\mathcal{F}^{-1} \frac{d}{dk} \mathcal{F}\right] \sim \frac{1}{k - i\kappa_{j,\ell}} \tag{39}$$

in the vicinity of $k = i\kappa_{j,\ell}$. The contribution of the resulting pole cancels against the explicit bound state contributions in eq. (35). (This is a general property of the phase shift approach.) As a result, eq. (35) can be recast to

$$\Delta E_\delta^{(N)} = -\frac{1}{2\pi} \int_m^\infty dt\, t \sum_\ell D_\ell \left[\nu_\ell(t)\right]_N \,. \tag{40}$$

[4] This also determines $\kappa = \frac{k}{E+m} = \frac{E-m}{k}$ as the genuine starting equation for the analytic continuation in k.

Here we have integrated by parts and also used the important property that the integral along the imaginary axis can be interchanged with the angular momentum sum [13]. A final change of variables $t \to \tau = \sqrt{t^2 - m^2}$ yields

$$\Delta E_\delta^{(N)} = -\frac{1}{2\pi} \int_0^\infty d\tau\, \tau \sum_\ell D_\ell \left[\nu_\ell(\sqrt{\tau^2 + m^2}) \right]_N . \tag{41}$$

Equation (41) is our master formula for the vacuum polarization energy of the string. However, we still have to make sense of the N subtractions.

B. Feynman Diagrams in $\overline{\text{MS}}$

Having subtracted the first N terms of the Born series to the vacuum polarization energy, we must add them back in form of Feynman diagrams. We then combine these diagrams with the counterterms, eq. (5), to renormalize the theory.

We generate the Feynman diagrams by expanding the effective action

$$\mathcal{A} = \mathcal{A}_0 - i\,\mathrm{Trln}\left\{ 1 + (i\partial\!\!\!/ - m)^{-1} H_I \right\} \tag{42}$$

with the interaction

$$H_I = L_\mu \gamma^\mu P_L + h + ip\gamma_5 \tag{43}$$

and the isospin operators

$$L_0 = 0\,, \qquad \vec{L} = 2\alpha_G \hat{\varphi} I_G(\varphi)\,, \qquad h = -\alpha_H \mathbb{1}\,, \qquad p = -\alpha_P I_P(\varphi)\,. \tag{44}$$

The isospin matrices I_G and I_P are defined in eq. (9). The first two orders of the expansion contain quadratic and sub–dominant logarithmic ultra–violet divergences. The third and fourth order terms are merely logarithmically divergent. The latter divergence can be extracted by omitting all external momenta in comparison with the loop momentum. These are the local contributions to the Feynman diagrams. To completely capture the divergence structure we therefore need to compute the first and second order completely (to control the quadratic divergences), but we may restrict ourselves to the local contribution in the case of the third and fourth order terms. This simplifies the computation considerably. In dimensional regularization the divergent contributions to the action from all four orders are (where $L \cdot L = L_\mu L^\mu$, etc.)

$$\begin{aligned}
\mathcal{A}^{(\mathrm{div})} \;=\; & i \int d^4x \left[\left(\frac{\mu}{m}\right)^{4-D} \int \frac{d^D l}{(2\pi)^D} \left(l^2 - 1 + i\epsilon\right)^{-2} \right] \\
& \times \mathrm{tr}_I \Bigg\{ \frac{1}{6} \left(\partial_\alpha L_\beta - i\,[L_\alpha, L_\beta]\right)^2 - \left[(h-m)^2 + p^2 \right] L \cdot L \\
& \quad - (\partial h)^2 - (\partial p)^2 + 2\,(h-m)\, L \cdot \partial p - iL \cdot [\partial p, p] \\
& \quad + 2m^2 \left[(h-m)^2 + p^2 - m^2 \right] + \left[(h-m)^2 + p^2 - m^2 \right]^2 \Bigg\} .
\end{aligned} \tag{45}$$

The counterterms in eq. (5) are identical to the integrand of the spatial integral.[5] Hence these divergences are removed by writing the counterterm coefficient as ($s = 1, \ldots, 4$)

$$c_s = -i \left(\frac{\mu}{m}\right)^{4-D} \int \frac{d^D l}{(2\pi)^D} \left(l^2 - 1 + i\epsilon\right)^{-2} + \overline{c}_s\,, \tag{46}$$

[5] For example, $\mathrm{tr}_I\,(D_\mu \Phi)^\dagger\,(D^\mu \Phi) = \frac{v^2}{m^2}\,\mathrm{tr}_I \big\{\,(\partial h)^2 + (\partial p)^2 + \left[(h-m)^2 + p^2\right] L \cdot L - 2\,(h-m)\, L \cdot \partial p + iL \cdot [\partial p, p]\,\big\}$.

where the $\overline{c}_s$ are the finite parts that are fixed by the renormalization conditions.

The remaining ultra–violet finite contribution from the first and second order is

$$\begin{aligned}\mathcal{A}^{(\mathrm{fin})} &= \frac{1}{(4\pi)^2}\int d^4x\,\mathrm{tr}_I\{4m^3h\} - \frac{1}{8\pi^2}\int\frac{d^4k}{(2\pi)^4}\left(m^2-\frac{k^2}{6}\right)\mathrm{tr}_I\left[h(k)h(-k)+p(k)p(-k)\right]\\ &\quad -\frac{1}{8\pi^2}\int\frac{d^4k}{(2\pi)^4}\int_0^1 dx\ln\left[1-x(1-x)\frac{k^2}{m^2}\right]\\ &\quad\times\mathrm{tr}_I\Big\{\left[m^2-x(1-x)k^2\right]\left[\tfrac{1}{2}L(k)\cdot L(-k)-3\left(h(k)h(-k)+p(k)p(-k)\right)\right]\\ &\qquad +x(1-x)\left[k\cdot L(k)\,k\cdot L(-k)-\tfrac{1}{2}k^2L(k)\cdot L(-k)\right]\\ &\qquad +2m^2p(k)p(-k)+imk\cdot L(k)p(-k)\Big\}, \end{aligned}\tag{47}$$

where the Fourier transforms have been indicated by the arguments of the profile functions. Observe that the m^3 and m^2 terms in the first two integrals combine to be proportional to an integral over $\mathrm{tr}_I\left[\Phi^\dagger\Phi-v^2\right]$. Hence they cancel exactly against the c_3–type counterterm if we set its finite part to $\overline{c}_3=\frac{1}{4\pi^2}\frac{m^4}{v^2}=\frac{f^2m^2}{4\pi^2}$. This defines the no–tadpole renormalization scheme, as the vacuum polarization energy is at least quadratic in the external fields. For the remaining counterterms, we discard all finite pieces,

$$\overline{c}_1=\overline{c}_2=\overline{c}_4=0\,.\tag{48}$$

This defines the $\overline{\mathrm{MS}}$ renormalization scheme that we impose further on. From eq. (47) we then determine the Feynman diagram contribution to the vacuum polarization energy up to quadratic order in the profiles to be

$$\begin{aligned}\Delta E_{\mathrm{FD}} &= \int_0^\infty\frac{kdk}{4\pi}\Big\{\frac{k^2}{3}\left(h_0^2+p_n^2\right)+4m^2I_1p_n^2+2mkI_1\alpha_c^{(+)}p_n+k^2I_2\left[\left(\alpha_c^{(+)}\right)^2-\left(\alpha_c^{(-)}\right)^2-(\alpha_s)^2\right]\\ &\qquad -\left(m^2I_1+k^2I_2\right)\left[6h_0^2+6p_n^2+\left(\alpha_c^{(+)}\right)^2+\left(\alpha_c^{(-)}\right)^2+(\alpha_s)^2\right]\Big\}.\end{aligned}\tag{49}$$

with the parameter integrals ($\eta=k/m$)

$$\begin{aligned}I_1 &= \int_0^1 dx\ln\left[1+x(1-x)\eta^2\right]=\frac{2}{\eta}\sqrt{4+\eta^2}\,\mathrm{arsinh}\left(\tfrac{\eta}{2}\right)-2\,,\\ I_2 &= \int_0^1 dx\,x(1-x)\ln\left[1+x(1-x)\eta^2\right]=\frac{\sqrt{4+\eta^2}}{3\eta^3}\left[\eta^2-2\right]\mathrm{arsinh}\left(\tfrac{\eta}{2}\right)+\frac{2}{3\eta^2}-\frac{5}{18}\,,\end{aligned}\tag{50}$$

and the Fourier transforms that contain the background profiles,

$$\begin{aligned}h_0(k) &= \int_0^\infty\rho\,d\rho\,\alpha_H(\rho)J_0(k\rho)\,,\qquad p_n(k)=\int_0^\infty\rho\,d\rho\,\alpha_P(\rho)J_n(k\rho)\,,\\ \alpha_c^{(\pm)}(k) &= \int_0^\infty\rho\,d\rho\,\alpha_G(\rho)c(\rho)\left[J_{n-1}(k\rho)\pm J_{n+1}(k\rho)\right]\,,\\ \alpha_s(k) &= \int_0^\infty\rho\,d\rho\,\alpha_G(\rho)s(\rho)J_1(k\rho)\,.\end{aligned}\tag{51}$$

We still have to work out the third and fourth order pieces, for which we want to avoid computing the full Feynman diagrams. Essentially we only need to compensate the logarithmic divergence

$$\mathcal{A}_{3,4}^{(\mathrm{div})}=\pi c_F\,TL\left[i\left(\frac{\mu}{m}\right)^{4-D}\int\frac{d^Dl}{(2\pi)^D}\left(l^2-1+i\epsilon\right)^{-2}\right]\tag{52}$$

where T and L are the (infinite) lengths of the time and z–axis intervals. The constant of proportionality is a simple integral over the profile functions,

$$c_F \;=\; \int_0^\infty \rho\, d\rho \Big\{ \left(\alpha_H^2+\alpha_P^2\right)\left(\alpha_H^2+\alpha_P^2+4m\alpha_H\right)+4\left(\alpha_H^2+\alpha_P^2+2m\alpha_H\right)\alpha_G^2 \\ -\frac{4n}{\rho}\,\alpha_G\alpha_P\left(s\alpha_P+c\alpha_H\right)\Big\}\,. \tag{53}$$

This result implies the limit, *cf.* eq. (40),

$$\sum_\ell D_\ell\, t\,[\nu_\ell(t)]_2 \;\stackrel{t\to\infty}{\longrightarrow}\; -\frac{c_F}{t}\,, \tag{54}$$

which is well suited to verify the accuracy of the numerical results.

C. Fake Boson

As already mentioned, dimensional analysis of the Dirac problem suggests that $N = 4$ subtractions are necessary in eqs. (35) and (41). This is unfortunate since the Feynman diagrams for $N = 3$ and $N = 4$ are complicated higher dimensional integrals (in Fourier space and over Feynman parameters). Also the corresponding Born orders are numerically costly when integrating the Dirac equation. As discussed above, we would like to employ an alternative method to deal with the logarithmic ultra–violet divergences that emerge at these orders. Here we will briefly explain one possibility. An important prerequisite for this, however, is the above analysis on the imaginary axis that allows us to perform the angular momentum sum *prior* to the momentum integral [13].

Essentially we do not want to subtract the third and fourth order terms of the Born series in eq. (41), but some other quantity that satisfies the following three conditions,

1) it must exhibit the same analytic properties as the $N = 3, 4$ terms of the Born series,

2) it must cancel the logarithmic divergence at large t: asymptotically it must behave like $\frac{c_F}{t}$,

3) we must be able to add it back as a sum of Feynman diagrams.

The perfect candidate that automatically satisfies conditions 1) and 3) is the second order contribution to the vacuum polarization energy of a background potential V coupled to a fluctuating boson field. To ensure the second condition, we merely need to fix the strength of the background potential such that the divergence of the corresponding Feynman diagram matches eq. (52). For definiteness, we take an exponential potential

$$V(\rho) = m^2\frac{\rho}{\rho_0}\,\mathrm{e}^{-2\rho/\rho_0}\,. \tag{55}$$

that also is translationally invariant along the z–axis. We choose the scale to be set by the return string because it determines the regions of momenta and angular momenta that dominate the integral and sum in eq. (41). We call $\overline{\nu}_\ell(t)$ the Jost function of this boson problem on the imaginary axis and $\overline{\nu}_\ell^{(2)}(t)$ its second Born approximation and define

$$\nu(t) = \lim_{\ell_{\max}\to\infty}\sum_{\ell=-n}^{\ell_{\max}} D_\ell\,[\nu_\ell(t)]_2 + \frac{c_F}{c_B}\lim_{\overline{\ell}_{\max}\to\infty}\sum_{\ell=0}^{\overline{\ell}_{\max}}(2-\delta_{\ell,0})\,\overline{\nu}_\ell^{(2)}(t)\,. \tag{56}$$

Here

$$c_B = \frac{1}{4}\int_0^\infty \rho\, d\rho\, V^2(\rho) = \frac{3m^4\rho_0^2}{512} \tag{57}$$

is the boson analogue to eq. (53). We have taken care to allow the (numerical) cut–offs for the angular momentum sums to be different in the fermion and boson cases. We merely have to ensure that, at a given t, either sum has converged to a sufficient accuracy.

Now we can perform the momentum integral and compute

$$\Delta E_\delta = -\frac{1}{2\pi}\int_0^\infty d\tau\, \tau\nu(\sqrt{\tau^2+m^2}) \tag{58}$$

as the phase shift contribution to the vacuum polarization energy.

Finally, we find the total vacuum polarization energy of the unwound string to be the sum of three terms

$$\Delta E = \Delta E_\delta + \Delta E_{\rm FD} + \Delta E_{\rm B}\,. \tag{59}$$

The last term originates from the finite part of the second order boson contribution,

$$\Delta E_{\rm B} = -\frac{c_F}{c_B}\int_0^\infty \frac{kdk}{16\pi}\, I_1 V_0^2\,, \tag{60}$$

with the Fourier transform

$$V_0(k) \;=\; \int_0^\infty \rho\, d\rho\, V(\rho) J_0(k\rho) = m^2\rho_0^2\, \frac{8-k^2\rho_0^2}{\left[4+k^2\rho_0^2\right]^{\frac{5}{2}}}\,. \tag{61}$$

Two remarks on the fake boson method are in order. First, it can be used with any renormalization prescription because the counterterms are determined from the two leading orders as a consequence of gauge invariance. Second, it is important to have identical masses for fake boson loop integral as in the coefficient of $\overline{\nu}_\ell^2$ in eq. (59). We note that this fake boson procedure has already been tested in similar models, including a consistent treatment of its mass parameter [22].

IV. RETURN STRING IN THE SINGULAR GAUGE

The techniques introduced so far allow us to compute the combined vacuum polarization energy of the physical and return strings, but we still have to disentangle them. In general this is a tedious or even impossible task. However, for the case that the return string is restricted to the chiral circle, the gauge–transformed Hamiltonian, eq. (12), suggests a potentially successful procedure. The pure return string is characterized by $f_G = f_H = 1$ which apparently leads to $\widetilde{H}_{\rm int} = 0$. This would actually imply that the return string has zero vacuum polarization energy. However, this conclusion is premature since eq. (12) has been derived for a constant angle ξ_1. Repeating the same gauge transformation but allowing a radial dependence as in eq. (13) yields

$$\widetilde{H} = -i\begin{pmatrix} 0 & \vec{\sigma}\cdot\hat{\rho} \\ \vec{\sigma}\cdot\hat{\rho} & 0 \end{pmatrix}\partial_\rho - \frac{i}{\rho}\begin{pmatrix} 0 & \vec{\sigma}\cdot\hat{\varphi} \\ \vec{\sigma}\cdot\hat{\varphi} & 0 \end{pmatrix}\partial_\varphi + m\beta + \frac{s'}{2c}\begin{pmatrix} -\vec{\sigma}\cdot\hat{\rho} & \vec{\sigma}\cdot\hat{\rho} \\ \vec{\sigma}\cdot\hat{\rho} & -\vec{\sigma}\cdot\hat{\rho} \end{pmatrix} I_P\,. \tag{62}$$

Here

$$s' = \frac{ds(\rho)}{d\rho} = \frac{\sin(\xi_1)}{1+\tanh(w_0)}\,\frac{w_0}{\rho_0}\,\left[\tanh^2\left(w_0\,\frac{\rho-\rho_0}{\rho_0}\right)-1\right]\,. \tag{63}$$

As a consequence of chiral symmetry, the gauge–transformed Hamiltonian only has an induced vector field while the (pseudo)scalar fields have disappeared.

In contrast to the gauge–transformed Hamiltonian of the physical string, eq. (12), no singularity emerges as $\rho \to 0$ and we have a well–defined scattering problem. It can be treated in complete analogy to what we described in previous sections using

$$\begin{aligned} \overline{\mathcal{M}}_{gg} &= \alpha_r \begin{pmatrix} R_- & 0 \\ 0 & R_- \end{pmatrix} \qquad \overline{\mathcal{M}}_{gf} = -z_\kappa \alpha_r \begin{pmatrix} 0 & R_- \\ R_- & 0 \end{pmatrix} \\ \overline{\mathcal{M}}_{ff} &= \alpha_r \begin{pmatrix} R_- & 0 \\ 0 & R_- \end{pmatrix} \qquad \overline{\mathcal{M}}_{fg} = z_\kappa^* \alpha_r \begin{pmatrix} 0 & R_- \\ R_- & 0 \end{pmatrix} , \end{aligned} \tag{64}$$

where $\alpha_r = \frac{s'}{2c}$ sets the order parameter of the corresponding Born series. Note that these are the coefficient matrices on for imaginary momenta, *i.e.* they are to be used in eq. (31).

The induced vector field only has a single isospin orientation, so the commutator term in the corresponding field tensor vanishes and only second order pieces contribute in eq. (45). As a result, the ultra–violet divergences from the third and fourth order Born terms vanish, which simplifies the renormalization procedure for the return string considerably. In particular, we do not need to apply the fake boson procedure.

For the return string we do not need to take special measures to impose the no–tadpole condition because the c_3–type counterterm vanishes on the chiral circle. Hence in the $\overline{\rm MS}$ scheme the vacuum polarization energy of the return string becomes

$$\Delta E^{(\rm r.s.)} = -\frac{1}{2\pi} \int_0^\infty d\tau\, \tau \sum_\ell D_\ell \left[\widetilde{\nu}_\ell(\sqrt{\tau^2 + m^2}) \right]_2 + \Delta E_{\rm FD}^{(\rm r.s.)} , \tag{65}$$

where $\widetilde{\nu}_\ell$ is the Jost function associated with the Hamiltonian, eq. (62), and

$$\Delta E_{\rm FD}^{(\rm r.s.)} = -\int \frac{kdk}{4\pi} \Big\{ m^2 I_1 \left[\left(\alpha_r^{(+)} \right)^2 + \left(\alpha_r^{(-)} \right)^2 \right] + 2k^2 I_2 \left(\alpha_r^{(-)} \right)^2 \Big\} \tag{66}$$

is the finite piece of the second order Feynman diagram. It includes the Fourier transforms

$$\alpha_r^{(\pm)}(k) = \int_0^\infty \rho d\rho\, \alpha_r(\rho) \left[J_{n+1}(k\rho) \pm J_{n-1}(k\rho) \right] . \tag{67}$$

We are now in a position to present numerical results.

V. NUMERICAL ANALYSIS

In this section we present results for the vacuum polarization energy

$$E_{\rm ps} = \Delta E - \Delta E^{(\rm r.s.)} \tag{68}$$

of the non–abelian string generated by the fluctuating fermions. The fermion mass $m = vf$ sets the scale for all data presented in this section. The length parameters $w_{G,H}$ and ρ_0 are measured in units of $\frac{1}{m}$, and since $E_{\rm ps}$ is the vacuum polarization energy per unit length of the string, it is measured in m^2. Note that the inverse width parameter (w_0) associated with the return string is dimensionless. As mentioned earlier, all results presented here are for unit winding of the string, $n = 1$.

The main purpose of the present study is to demonstrate, as a proof–of–principle, that the leading (fermion) contribution to the vacuum polarization energy of a string in a non-abelian

ρ_0	6	8	10	12	14
ϵ_0	0.0065	0.0024	0.0014	0.0010	0.0008

TABLE I: Lowest bound state energy for $\xi_1 = \frac{\pi}{2}$, $w_H = w_G = 2$ and $w_0 = 8$ as a function of the position of the return string, ρ_0.

gauge theory can indeed be unambiguously computed. This has made necessary the introduction of a return string. We parameterized its shape by two variables, the position ρ_0 and the (inverse) width w_0. In this study we concentrate on the variation of $E_{\rm ps}$ on these two parameters, to establish a well–defined limit. The underlying feature of our calculations is the separation of the physical and return strings' vacuum energies. Before presenting the corresponding results, we like to discuss a necessary condition for this separation to be observed. The physical string generates a zero mode for $\xi_1 = \frac{\pi}{2}$ in the $\ell = -n$ channel [5], regardless of the specific shape that is parameterized by w_H and w_G. This zero mode must also be produced when the physical string is augmented by the return string if the separation hypothesis holds, because the bound state wave function does not interfere with the return string. In table I we present the lowest single particle energy[6] for $\xi_1 = \frac{\pi}{2}$ as a function of the separation between the physical and the return string. As expected, this eigenvalue approaches zero as the separation increases. These results provide confidence in the separation hypothesis already at moderate values of ρ_0.

The numerical calculation of the vacuum polarization energy is quite involved. The main effort concerns the phase shift part, ΔE_δ in eq. (59). In addition to the criteria mentioned after eqs. (21) and (54) we have furthermore tested these calculations of scattering data with respect to

- charge conjugation symmetry: $E \to -E$ (for real momenta)
- reflection around the winding of the string: $\ell \to -\ell - n$
- sum rules for distinct angular momenta, *i.e.* agreement of integrals involving the Jost–function over real and imaginary momenta.

We integrate the differential eqs. (23), (31), their Born expansions as well as the fake boson and return string analogues from some large radius $\rho_{\rm max} \sim 4\rho_0$ to $\rho_{\rm min} \sim 0$ with the boundary condition $\mathcal{F}(\rho_{\rm max}, k) = \mathbf{1}$, and identify $\lim_{\rho\to 0} \mathcal{F}(\rho, k) = \mathcal{F}(\rho_{\rm min}, k)$. Alternatively, this identification can equally well be obtained from the derivative of the wave–function. Furthermore, a differential equation is formulated for $\ln\det \mathcal{F}(\rho,k)^{-1}\mathcal{F}(\rho,k)^*$ to avoid 2π ambiguities in the computation of the phase shift, $\delta_\ell(k)$, *cf.* eq (36). The computations for real momenta have been performed mainly for use in the consistency tests mentioned above. Channels that include (modified) Hankel functions with zero index ($\ell = -2, -1, 0$) are particularly cumbersome because in separating the regular and irregular solutions, we need to distinguish $\ln(\rho)$ from a constant at very small distances $\rho_{\rm min}$, *cf.* eq. (22). That is, $\rho_{\rm min}$ must be taken tiny to obtain the correct scattering matrix in eq. (21). On the real axis, the result can be checked against extracting the $\mathcal{S}$–matrix from the derivative of the scattering wave–function. For calculations on the imaginary axis we assume $\rho_{\rm min} \sim 10^{-60}$ and successively carry out an extrapolation

$$\nu(\rho_{\rm min}) = \nu_0 + \frac{a_1}{\ln(\rho_{\rm min})} + \frac{a_2}{\ln^2(\rho_{\rm min})} \cdots \ , \tag{69}$$

[6] We compute this energy by discretizing the spectrum of the free Dirac Hamiltonian in a large cylindrical cavity, computing the interaction matrix elements from those eigenstates, and diagonalizing the resulting matrix.

w_0	ρ_0	$\Delta E_\delta^{(\mathrm{r.s.})}$	$\Delta E_{\mathrm{FD}}^{(\mathrm{r.s.})}$	$\Delta E^{(\mathrm{r.s.})}$
8	6	-0.006	-0.100	-0.107
8	8	-0.004	-0.048	-0.052
8	10	-0.003	-0.027	-0.030
8	12	-0.002	-0.018	-0.020
8	14	-0.002	-0.012	-0.014
6	6	-0.003	-0.037	-0.040
6	8	-0.002	-0.018	-0.020
6	10	-0.001	-0.011	-0.013
6	12	-0.001	-0.007	-0.008
6	14	-0.001	-0.005	-0.006

TABLE II: Numerical results for the vacuum polarization energy of the return string in the $\overline{\mathrm{MS}}$ scheme. The entry $\Delta E_\delta^{(\mathrm{r.s.})}$ refers to the τ integral in eq. (65). It is the analogue of eq. (58) for the sole return string. Scales are set by the fermion mass m.

for the Jost function in these channels. We test the final result, *i.e.* ν_0, for stability against further changes of $\rho_{\min}$ and also check the condition $\mathsf{Im}(\nu_0) = 0$. In other channels $\rho_{\min} \sim 10^{-12}$ suffices to represent the origin.

For the sum over angular momentum channels, we must go to very large channel numbers (typically several hundred). The required value $\ell_{\max}$ to obtain convergence increases with ρ_0. While this is expected, since ρ_0 determines the impact parameter, it limits our ability to send $\ell_{\max}$ to infinity. We therefore perform an extrapolation to $\ell_{\max} \to \infty$. Using various analytical forms for this extrapolation we estimate an absolute error of 0.005 for ΔE_δ. Fortunately, the right hand side of eq. (56) approaches zero at moderate momenta, so that we can approximate the upper limit of the integral in eq. (58) by 10 or even less.

The individual sums in eq. (56) have typical maximal values of the order of tens, while they combine to a result at the order of a tenth or less. This behavior is typical for these type of computations because the physics information is hidden beneath the ultra–violet divergences. This disproportion increases with the distance of the return string. For practical purposes, this unfortunately implies a severe loss of precision of three orders of magnitude or more. However, demanding too high numerical precision makes the integration of the Dirac equation (and its Born expansion) ineffective. Fortunately, doing the momentum integral on the imaginary axis regains quite some efficency because we are not plagued by oscillating integrands.

In total, it takes several CPU–days on a modern PC to generate a number for ΔE for a single set of parameters $(w_{G,H}, \xi_1, w_0, \rho_0)$. The fake boson piece $\sum_\ell \overline{\nu}_\ell^2$, in comparison, is cheap since it takes less than a CPU–hour even though $\overline{\ell}_{\max} \approx 3\ell_{\max}$ for sufficient convergence.

We first consider the return string contribution, which only depends on the parameters w_0 and ρ_0. The results are shown in table II. We observe that $\Delta E^{(\mathrm{r.s.})}$ is small in magnitude and quickly decreases as the position of the return string is sent to infinity. Dimensional analysis indicates that except for the Higgs kinetic term (the c_2–type in eq. (5)) all contributions from the return string vanish as $\rho_0 \to \infty$. This is the local part of the second order Feynman diagram, which is taken out in the $\overline{\mathrm{MS}}$ scheme. Hence both the smallness and the large ρ_0 behavior of $\Delta E^{(\mathrm{r.s.})}$ are well understood. Since for our particular parameterization, the relevant parameter is w_0/ρ_0, the above argument also explains the decrease with w_0 that can be observed from table II. In essence, the contribution of the sole return string is well under control.

In table III we show the contributions to the vacuum polarization energy of the configuration consisting of both the physical and the return string, for various parameters that describe the

w_0	ρ_0	ΔE_δ	$\Delta E_{\rm FD}$	$\Delta E_{\rm B}$	ΔE
8	6	0.168	-0.120	0.013	0.061
8	8	0.153	-0.105	0.012	0.060
8	10	0.144	-0.096	0.013	0.061
8	12	0.132	-0.090	0.013	0.055
8	14	0.121	-0.085	0.013	0.049
6	6	0.148	-0.080	0.012	0.080
6	8	0.137	-0.072	0.012	0.077
6	10	0.128	-0.067	0.011	0.072
6	12	0.118	-0.064	0.012	0.066
6	14	0.107	-0.061	0.012	0.057

TABLE III: Numerical results for the vacuum polarization energy of the combination of the physical and the return strings in the $\overline{\rm MS}$ scheme with $w_H = w_G = 2$ and $\xi_1 = 0.4\pi$.

w_0	ρ_0	ΔE	$-\Delta E^{(\rm r.s.)}$	$\Delta E_{\rm ps}$
8	6	0.061	0.107	0.168
8	8	0.060	0.052	0.112
8	10	0.061	0.030	0.091
8	12	0.055	0.020	0.075
8	14	0.049	0.013	0.062
6	6	0.080	0.040	0.120
6	8	0.077	0.020	0.097
6	10	0.072	0.014	0.086
6	12	0.066	0.008	0.074
6	14	0.057	0.006	0.063

TABLE IV: Vacuum polarization energy of the physical string for $w_H = 2$, $w_G = 2$ and $\xi_1 = 0.4\pi$.

return string part of the configuration. We observe a significant cancellation between the integral over the subtracted Jost–function and the fermion Feynman diagram contribution. This behavior, of course, can be different in renormalization schemes other than $\overline{\rm MS}$. The fake boson Feynman diagram contribution is negligible in the $\overline{\rm MS}$ scheme.

In table IV we combine the results from the previous two tables to find the vacuum polarization energy of the physical string. Certainly the data indicate a saturation for large ρ_0, as demanded by the assumption that the scales for the physical and return strings separate This separation assumption is further supported by the observation that for increasing distance of the return string, its width ($\frac{1}{w_0}$) no longer affects the energy of the physical string. In this limit, the vacuum polarization energy of the string turns out to be less than a tenth of the fermion mass. We have repeated this analysis for a thin string and present the results in table V. We find that its vacuum polarization energy is even smaller.

We recall that for typical model parameters, *i.e.* gauge and Yukawa couplings[7] of the order unity or less, the classical energy of the physical string configuration, which we did not discuss here, is of the order 1 or 10. Also typical single particle energies of the fermions bound in the string background are around 0.5. Hence the vacuum polarization energy in the $\overline{\rm MS}$ renormalization

[7] The relevant scale is set by the heaviest fermion coupling to the string, *i.e.* the top quark, where $f \sim 175/190 \sim 1$ as the Weinberg angle is set to zero.

ρ_0	ΔE	$-\Delta E^{(\mathrm{r.s.})}$	ΔE_{ps}
6	0.003	0.107	0.110
8	0.022	0.052	0.074
10	0.029	0.030	0.059
12	0.027	0.020	0.047
14	0.025	0.013	0.038

TABLE V: Same as table IV for $w_H = 0.5$, $w_G = 1$ and $\xi_1 = 0.4\pi$. Here we used $w_0 = 8$.

scale is about an order of magnitude smaller than other relevant energy scales in the model. This smallness of string (vortex) vacuum polarization energies has also been observed for string type configurations in simpler models [8] and hence is not fully unexpected.

We have also performed numerical calculations with the singular Hamiltonian of eq. (12) by regularizing the singularity of the gauge coupling: $\frac{1}{\rho} \to \frac{1}{\rho + w_{\mathrm{reg}}}$. In the second order scattering problem the unregularized form induces a $\frac{1}{\rho^2}$ divergence at small ρ, which eventually spoils the analytic properties of the scattering data. However, we did not obtain a smooth limit as we removed the regulator. This is not a contradiction to the existence of a well defined vacuum polarization energy. It just implies that two operations of (i) integrating to $\rho \to 0$ and (ii) removing the regulator may not be exchanged. This can already be observed from computing the classical energy with such a regulator. The spatial integral, which involves the field strength tensor squared, has no well–defined limit as the regulator w_{reg} is sent to zero.

Finally, we would like to mention that we have also considered the numerical computation on the real axis, *e.g.* using eq. (35), to compute the vacuum polarization energy for this Dirac problem. As expected from simpler models, we find that a twofold subtraction suffices to render a convergent integral when the angular momentum sum is carried out first. This result contradicts the ultra–violet divergence structure of the quantum field theory. It is a mathematical artifact stemming from the improper exchange of sum and integrals [13]. This has been the major reason to analytically continue to imaginary momenta, since it allows this exchange. We remark that a rigorous Born subtraction, *i.e.* up to at least fourth order in $\overline{\mathcal{M}}_i$ in eq. (23), can indeed be carried out on the real momentum axis. As argued before this seems technically problematic. It is the exchange of sums and integrals that requires absolute convergence, which is only guaranteed for imaginary momenta.

VI. CONCLUSIONS

We have presented a feasible method for the computation of the fermion contribution to the vacuum polarization energy of a string type configuration in a non–abelian gauge theory. If there are many internal degrees of freedom for the fermions, for example color, this contribution is the dominant quantum correction to the energy of the string. The key tool that we employed was the introduction of a *return string* configuration on the chiral circle. It was, in particular, indispensable to obtain a well–defined Feynman series, which is required to impose conventional renormalization conditions. For the present study we used the $\overline{\mathrm{MS}}$ scheme, augmented by the no–tadpole condition. The generalization to other schemes, such as on–shell, does not cause further (technical) subtleties. The advantage of having the return string on the chiral circle is that its vacuum polarization energy can be reliably computed in a specific gauge. Then it suffices to compute the energy of any physical string configuration with respect to this return string. Furthermore, technical innovations such as the computation of momentum integrals via analytic continuation and the introduction of a fake boson field have been necessary to reduce the computational workload, making the investigation

feasible with appropriate expenditure of computational resources.

Our results suggest that for future investigations of the total energy of string configurations it is a good approximation to omit the $\overline{\rm MS}$ contribution to the vacuum polarization energy. This contribution is about an order of magnitude smaller than other typical energies in the model. This is in particular the case for a thin string. Such an approximation simplifies the computation of the energy of the string tremendously: It merely remains to find the (finite pieces of) the counterterm coefficients, since the spatial integral of these terms is, by the definition of renormalizability, the same as for the classical energy. The remaining spatial integrals only involve gauge invariant combinations and thus can immediately be computed *without* the return string. This result will make the search for configurations feasible that are stabilized on purely dynamical grounds. For instance, such a stabilization can occur by trapping fermions along the string. Once such a configuration is discovered, it remains to verify that its vacuum polarization energy in the $\overline{\rm MS}$ scheme indeed is tiny, which can be achieved with the techniques presented here. We leave this study to a forth–coming paper.

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